Feedback stabilization of the wake behind a steady cylinder

J. Pralits[1], T. Bewley[2] and P. Luchini[1]

[1]DIMEC, Università di Salerno, Fisciano (SA), Italy

[2]Flow Control Lab, Dept. of MAE, UC San Diego, USA

7th ERCOFTAC SIG33 - FLUBIO WORKSHOP ON OPEN ISSUES IN TRANSITION AND FLOW CONTROL
S. Margherita Ligure 16-18 October 2008

Motivation

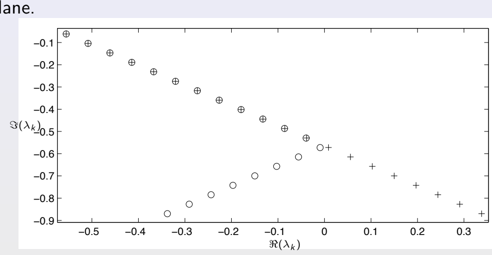
- Optimal control of wake instabilities via application of modern control algorithms (Riccati equation) is intractable because of the very large number of degrees of freedom deriving from the discretization of the Navier-Stokes equations.
- The research approach until today has been to use eg. reduced-order models (ROM)
- Here we show an approach based on direct and adjoint eigenvectors which make, at least in some cases, mathematically rigorous optimal control a reality.

Framework

- The Salerno group has experience in the computation and use of direct and adjoint modes of large-scale recirculating flows, linearized about unstable equilibria.
- The UCSD group has developed an efficient technique to compute minimal-energy stabilizing linear feedback control rules for linear systems. This technique is based solely on the unstable eigenvalues and corresponding left eigenvectors of the linearized open-loop system.

Overview

If a minimal-energy stabilizing feedback rule $\mathbf{u} = \mathbf{Kx}$ is applied to the system $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$, the eigenvalues of the closed-loop system $\mathbf{A} + \mathbf{BK}$ are given by the union of the stable eigenvalues of $\mathbf{A}$ and the reflection of the unstable eigenvalues of $\mathbf{A}$ into the left-half plane.



- Since we know where the closed-loop eigenvalues of the system are, the feedback gain matrix **K** in this problem may be computed by the process of pole assignment
- Applying this process to the equation governing the dynamics of the system in modal form, and then transforming appropriately, leads to an expression for **K** requiring only the knowledge of the unstable modes, as shown in the following

The linear optimal control problem

The classical full-state-information control problem is formulated as: for the state **x** and the control **u** related via the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \text{on} \quad 0 < t < T \quad \text{with} \quad \mathbf{x} = \mathbf{x}_0 \quad \text{at} \quad t = 0$$

find the control **u** that minimizes the cost function

$$J = \frac{1}{2}\int_0^T [\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}]\, dt.$$

The adjoint variable **r** is introduced as a Lagrange multiplier. The variations of the augmented cost function

$$J = \int_0^T \frac{1}{2}[\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}] + \mathbf{r}^*[\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{u}]\, dt.$$

gives $\dot{\mathbf{r}} = -\mathbf{A}^H\mathbf{r} - \mathbf{Q}\mathbf{x}, \quad \mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^H\mathbf{r} \quad \text{with} \quad \mathbf{r}(t = T) = 0$

A boundary-value problem

The state and adjoint equations may be written in the combined matrix form

$$\frac{d\mathbf{z}}{dt} = \mathbf{Z}\mathbf{z} \quad \text{where} \quad \mathbf{Z} = \mathbf{Z}_{2n\times 2n} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H \\ -\mathbf{Q} & -\mathbf{A}^H \end{bmatrix} \qquad (1)$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{r} \end{bmatrix}, \quad \text{and} \quad \begin{cases} \mathbf{x} = \mathbf{x}_0 & \text{at} \quad t = 0, \\ \mathbf{r} = 0 & \text{at} \quad t = T. \end{cases}$$

($\mathbf{Z}$ has a Hamiltonian symmetry, such that eigenvalues appear in pairs of equal imaginary and opposite real part.) This linear ODE is a two-point boundary value problem and may be solved assuming there exist a relationship between the state vector $\mathbf{x}(t)$ and adjoint vector $\mathbf{r}(t)$ vi a matrix $\mathbf{X}(T)$ such that $\mathbf{r} = \mathbf{X}\mathbf{x}$, and inserting this solution ansatz into (1) to eliminate $\mathbf{r}$.

The Riccati equation

It follows that matrix $\mathbf{X}$ obeys the differential Riccati equation

$$-\frac{d\mathbf{X}}{dt} = \mathbf{A}^H\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H\mathbf{X} + \mathbf{Q} \quad \text{where} \quad \mathbf{X}(t = T) = 0. \qquad (2)$$

Once $\mathbf{X}$ is known, the optimal value of $\mathbf{u}$ may then be written in the form of a feedback control rule such that

$$\mathbf{u} = \mathbf{K}\mathbf{x} \quad \text{where} \quad \mathbf{K} = -\mathbf{R}^{-1}\mathbf{B}^H\mathbf{X}.$$

Finally, if the system is time invariant and we take the limit that $T \to \infty$, the matrix $\mathbf{X}$ in (2) may be marched to steady state. This steady state solution for $\mathbf{X}$ satisfies the continuous-time algebraic Riccati equation

$$0 = \mathbf{A}^H\mathbf{X} + \mathbf{X}\mathbf{A} - \mathbf{X}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H\mathbf{X} + \mathbf{Q},$$

where additionally $\mathbf{X}$ is constrained such that $\mathbf{A} + \mathbf{B}\mathbf{K}$ is stable.

The classical way of solution

A linear time-invariant system can be solved using its eigenvectors. Assume that an eigenvector decomposition of the $2n \times 2n$ matrix $\mathbf{Z}$ is available such that

$$\mathbf{Z} = \mathbf{V}\boldsymbol{\Lambda}_c\mathbf{V}^{-1} \quad \text{where} \quad \mathbf{V} = \begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{z} = \begin{bmatrix} \mathbf{x} \\ \mathbf{r} \end{bmatrix}$$

and the eigenvalues of $\mathbf{Z}$ appearing in the diagonal matrix $\boldsymbol{\Lambda}_c$ are enumerated in order of increasing real part. Since

$$\mathbf{z} = \mathbf{V}e^{\boldsymbol{\Lambda}_c t}\mathbf{V}^{-1}\mathbf{z}_0$$

the solutions $\mathbf{z}$ that obey the boundary conditions at $t = \infty$ are spanned by the first n columns of $\mathbf{V}$. The direct ($\mathbf{x}$) and adjoint ($\mathbf{r}$) parts of the these columns are related as $\mathbf{r} = \mathbf{X}\mathbf{x}$, where

$$\mathbf{X} = \mathbf{V}_{21}\mathbf{V}_{11}^{-1}$$

The minimal-energy stabilizing feedback control

Taking the limit as $\mathbf{Q} \to 0$ (maintain constraint that $\mathbf{x}^*\mathbf{Q}\mathbf{x}$ should be integrable), we obtain the so called *minimal-energy stabilizing feedback control*. In this limit $\mathbf{Z}$ becomes block triangular, and the direct and adjoint equations become

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^H\mathbf{r}, \quad \dot{\mathbf{r}} = -\mathbf{A}^H\mathbf{r} \qquad (3)$$

The minimal-energy stabilizing feedback control

Taking the limit as $\mathbf{Q} \to 0$ (maintain constraint that $\mathbf{x}^*\mathbf{Qx}$ should be integrable), we obtain the so called *minimal-energy stabilizing feedback control*. In this limit $\mathbf{Z}$ becomes block triangular, and the direct and adjoint equations become

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \quad \mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^H\mathbf{r}, \quad \dot{\mathbf{r}} = -\mathbf{A}^H\mathbf{r} \tag{3}$$

The eigenvalues of this system is given by the union of the eigenvalues of $\mathbf{A}$ and the eigenvalues of $-\mathbf{A}^H$.

Introduction The optimal control problem Min-energy feedback control Results

The minimal-energy stabilizing feedback control

Taking the limit as $\mathbf{Q} \to 0$ (maintain constraint that $\mathbf{x}^*\mathbf{Qx}$ should be integrable), we obtain the so called *minimal-energy stabilizing feedback control*. In this limit $\mathbf{Z}$ becomes block triangular, and the direct and adjoint equations become

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \quad \mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^H\mathbf{r}, \quad \dot{\mathbf{r}} = -\mathbf{A}^H\mathbf{r} \tag{3}$$

The eigenvalues of this system is given by the union of the eigenvalues of $\mathbf{A}$ and the eigenvalues of $-\mathbf{A}^H$.
Denoting:
$\mathbf{x}^i$ and λ^i the i-th right eigenvector and eigenvalue of $\mathbf{A}$,
$\mathbf{y}^i$ and $-\lambda^{i*}$ the i-th right eigenvector and eigenvalue of $-\mathbf{A}^H$ ($\mathbf{y}^{i*}$ is left e.v. of $\mathbf{A}$), we see that the stable eigenvectors of (3) are of two possible types:

The minimal-energy stabilizing feedback control

Taking the limit as $\mathbf{Q} \to 0$ (maintain constraint that $\mathbf{x}^*\mathbf{Q}\mathbf{x}$ should be integrable), we obtain the so called *minimal-energy stabilizing feedback control*. In this limit $\mathbf{Z}$ becomes block triangular, and the direct and adjoint equations become

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^H\mathbf{r}, \quad \dot{\mathbf{r}} = -\mathbf{A}^H\mathbf{r} \tag{3}$$

The eigenvalues of this system is given by the union of the eigenvalues of $\mathbf{A}$ and the eigenvalues of $-\mathbf{A}^H$.
Denoting:
$\mathbf{x}^i$ and λ^i the i-th right eigenvector and eigenvalue of $\mathbf{A}$,
$\mathbf{y}^i$ and $-\lambda^{i*}$ the i-th right eigenvector and eigenvalue of $-\mathbf{A}^H$
($\mathbf{y}^{i*}$ is left e.v. of $\mathbf{A}$), we see that the stable eigenvectors of (3) are of two possible types:

$\mathbf{r} = 0,\ \mathbf{x} = \mathbf{x}^i$ if $\Re(\lambda^i) < 0$ (stable)

Introduction | The optimal control problem | Min-energy feedback control | Results

The minimal-energy stabilizing feedback control

Taking the limit as $\mathbf{Q} \to 0$ (maintain constraint that $\mathbf{x}^*\mathbf{Q}\mathbf{x}$ should be integrable), we obtain the so called *minimal-energy stabilizing feedback control*. In this limit $\mathbf{Z}$ becomes block triangular, and the direct and adjoint equations become

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^H\mathbf{r}, \quad \dot{\mathbf{r}} = -\mathbf{A}^H\mathbf{r} \tag{3}$$

The eigenvalues of this system is given by the union of the eigenvalues of $\mathbf{A}$ and the eigenvalues of $-\mathbf{A}^H$.
Denoting:
$\mathbf{x}^i$ and λ^i the i-th right eigenvector and eigenvalue of $\mathbf{A}$,
$\mathbf{y}^i$ and $-\lambda^{i*}$ the i-th right eigenvector and eigenvalue of $-\mathbf{A}^H$
($\mathbf{y}^{i*}$ is left e.v. of $\mathbf{A}$), we see that the stable eigenvectors of (3) are of two possible types:

$\mathbf{r} = 0,\ \mathbf{x} = \mathbf{x}^i$ if $\Re(\lambda^i) < 0$ (stable)
$\mathbf{r} = \mathbf{y}^i,\ \mathbf{x} = (\lambda^{i*} + \mathbf{A})^{-1}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H\mathbf{y}^i$ if $\Re(\lambda^i) > 0$ (unstable)

We now project an arbitrary initial condition $\mathbf{x}_0$ onto these modes,

$$\mathbf{x}_0 = \sum_{\text{stable}} d_j \mathbf{x}^j + \sum_{\text{unstable}} e_j (\lambda^{j*} + \mathbf{A})^{-1} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^H \mathbf{y}^j \qquad (4)$$

and note that in order to reconstruct $\mathbf{r}$ we only need the e_j's, because the stable modes have $\mathbf{r} = 0$.

Introduction The optimal control problem **Min-energy feedback control** Results

We now project an arbitrary initial condition $\mathbf{x}_0$ onto these modes,

$$\mathbf{x}_0 = \sum_{\text{stable}} d_j \mathbf{x}^j + \sum_{\text{unstable}} e_j (\lambda^{j*} + \mathbf{A})^{-1} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^H \mathbf{y}^j \qquad (4)$$

and note that in order to reconstruct $\mathbf{r}$ we only need the e_j's, because the stable modes have $\mathbf{r} = 0$.The coefficients d_j can be eliminated from (4) by projecting the left eigenvectors:

$$\mathbf{y}^{i*} \mathbf{x}_0 = \mathbf{y}^{i*} \sum_{\text{unstable}} e_j (\lambda^{j*} + \mathbf{A})^{-1} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^H \mathbf{y}^j = \sum_{\text{unstable}} f_{ij} e_j$$

where, since $\mathbf{y}^{i*}$ is also a left eigenvector of $(\lambda^{j*} + \mathbf{A})^{-1}$,

$$f_{ij} = \frac{\mathbf{y}^{i*} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^H \mathbf{y}^j}{\lambda^i + \lambda^{j*}}$$

We now project an arbitrary initial condition $\mathbf{x}_0$ onto these modes,

$$\mathbf{x}_0 = \sum_{\text{stable}} d_j \mathbf{x}^j + \sum_{\text{unstable}} e_j (\lambda^{j*} + \mathbf{A})^{-1} \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H \mathbf{y}^j \qquad (4)$$

and note that in order to reconstruct $\mathbf{r}$ we only need the e_j's, because the stable modes have $\mathbf{r} = 0$.The coefficients d_j can be eliminated from (4) by projecting the left eigenvectors:

$$\mathbf{y}^{i*}\mathbf{x}_0 = \mathbf{y}^{i*} \sum_{\text{unstable}} e_j (\lambda^{j*} + \mathbf{A})^{-1} \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H \mathbf{y}^j = \sum_{\text{unstable}} f_{ij} e_j$$

where, since $\mathbf{y}^{i*}$ is also a left eigenvector of $(\lambda^{j*} + \mathbf{A})^{-1}$,

$$f_{ij} = \frac{\mathbf{y}^{i*}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^H\mathbf{y}^j}{\lambda^i + \lambda^{j*}}$$

Only the unstable eigenvalues and left eigenvectors are needed.

The main theorem

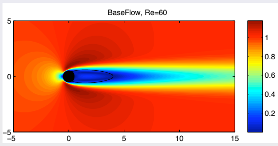
Summarizing, the solution of the minimal-energy stabilizing control feedback problem can be written in terms of the unstable left eigenvectors only.

Theorem 1. *Consider a stabilizable system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ with no pure imaginary open-loop eigenvalues. Determine the unstable eigenvalues and corresponding left eigenvectors of $\mathbf{A}$ such that $\mathbf{T}_u^H\mathbf{A} = \mathbf{\Lambda}_u\mathbf{T}_u^H$ (equivalently, determine the unstable eigenvalues and corresponding right eigenvectors of $\mathbf{A}^H$ such that $\mathbf{A}^H\mathbf{T}_u = \mathbf{T}_u\mathbf{\Lambda}_u^H$). Define $\bar{\mathbf{B}}_u = \mathbf{T}_u^H\mathbf{B}$ and $\mathbf{C} = \bar{\mathbf{B}}_u\bar{\mathbf{B}}_u^H$, and compute a matrix $\mathbf{F}$ with elements $f_{ij} = c_{ij}/(\lambda_i + \lambda_j^*)$. The minimal-energy stabilizing feedback controller is then given by $\mathbf{u} = \mathbf{K}\mathbf{x}$, where $\mathbf{K} = -\bar{\mathbf{B}}_u^H\mathbf{F}^{-1}\mathbf{T}_u^H$.*

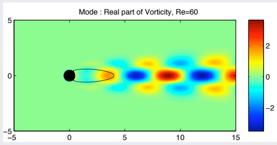
Application

Computation of minimal-energy stabilizing linear feedback control to suppress vortex shedding from a circular cylinder ($Re = UD/\nu$)

- Full state information
- Actuator: rotational oscillation
- One pair of unstable complex conjugate modes
- $\underbrace{\mathbf{K}}_{1\times n} = -\underbrace{\bar{\mathbf{B}}_u^H}_{1\times 2}\underbrace{\mathbf{F}^{-1}}_{2\times 2}\underbrace{\mathbf{T}_u^H}_{2\times n}$, real valued



mean velocity field modulus



real part of vorticity (unstable)

Background: control using rotational oscillation

Aim: reduce C_D

Exp. Tokumaru & Dimotakis (1991), -20%, $Re = 15000$

Feedback control:

Exp. Fujisawa & Nakabayashi (2002) -16% (-70% C_L), $Re = 20000$

Exp. Fujisawa et al.(2001) "reduction", $Re = 6700$

Optimal control (using adjoints):

Num. He et al.(2000) -30 to -60% for $Re = 200 - 1000$

Num. Protas & Styczek (2002) -7% at $Re = 75$, -15% at $Re = 150$

Num. Bergmann et al.(2005) -25% at $Re = 200$ (POD)

Aim: reduce vortex shedding

Feedback control:

Num. Protas (2004) reduction, "point vortex model", $Re = 75$

Optimal control (using adjoints):

Num. Homescu et al.(2002) reduction, $Re = 60 - 1000$

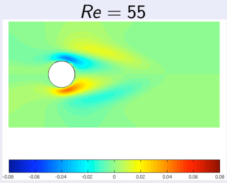
Numerical procedure

- All equations are discretized using second-order finite-differences over a staggered, stretched, Cartesian mesh.
- An immersed-boundary technique is used to enforce the boundary conditions on the cylinder.
- The system of algebraic equations deriving from the disretization of the nonlinear mean-flow equations, along with their boundary conditions, is solved by a Newton-Raphson procedure.
- The eigenvalue problem is solved by inverse iteration, both right and left eigenvectors are solved simultaneously, as in the work by Giannetti & Luchini[1]
- The linear and nonlinear evolution equations are solved using Adams-Bashforth/Crank-Nicholson

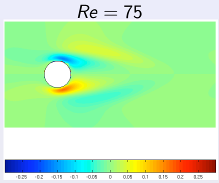
[1]Structural sensitivity of the first instability of the cylinder wake, J. Fluid Mech. **581**, 167 (2007)

Results: K_u

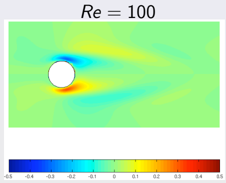
$Re = 55$



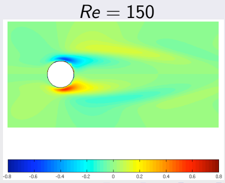
$Re = 75$



$Re = 100$

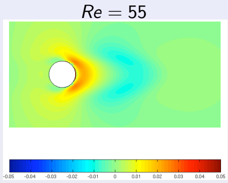


$Re = 150$

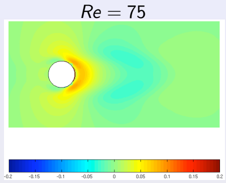


Results: K_v

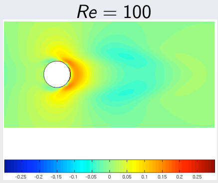
$Re = 55$



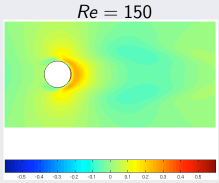
$Re = 75$



$Re = 100$

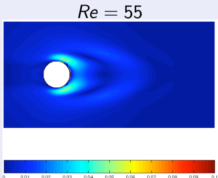


$Re = 150$

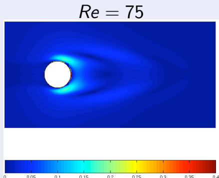


Results: $|\mathbf{K}|$

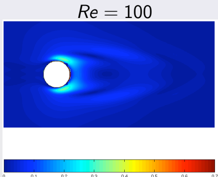
$Re = 55$



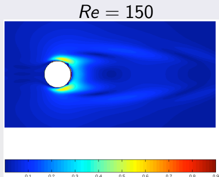
$Re = 75$



$Re = 100$



$Re = 150$



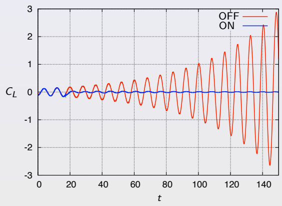
Results: linearized N-S equations

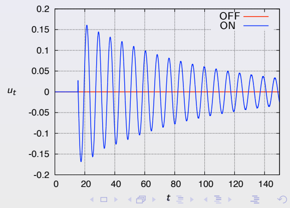
Applying the control to the linearized system allows us to check if "pole assignment" actually works.

- With the control OFF: $u \sim \exp(i\lambda_i t)\exp(\lambda_r t)$
- With the control ON: $u \sim \exp(i\lambda_i t)\exp(-\lambda_r t)$

C_L: lift coefficient; u_t tangential velocity of the cylinder.

Test case: $Re = 55$, control is turned on at $t = 18$





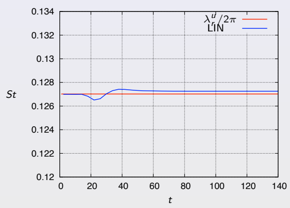
Introduction The optimal control problem Min-energy feedback control Results

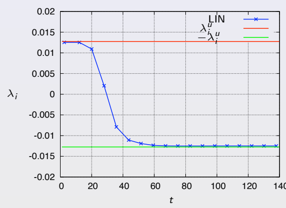
Results: linearized N-S equations

The temporal evolution of the frequency and growth rate is compared with the eigenvalue λ

- The Strouhal number: $St = fD/U$ compared to $St = \lambda_r/2\pi$
- The growth rate: $\sigma = \frac{d}{dt}log(u(t))$ compared to λ_i

Test case: $Re = 55$, control is turned on at $t = 18$





Results: N-S equations

Vorticity: No Control | Vorticity: With Control

Screen

NoControl

Screen

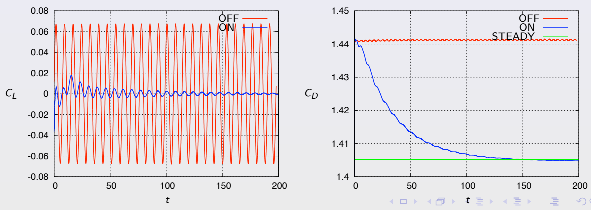
WithControl

Results: N-S equations

The temporal evolution of the lift (C_L) and drag (C_D) coefficients with and without control

- C_L goes to zero as the control is applied
- C_D goes towards value for steady state solution (-2.5%)
- Control, so far, verified to work up to $Re = 75$

Test case: $Re = 55$, control is turned on at $t = 0$



Conclusions

- The solution of the minimal-energy stabilizing control feedback problem can be written in terms of the unstable left eigenvectors only.
- A practical algorithm to do so has been devised and tested on the cylinder wake.
- An optimal controller using rotational oscillations as actuator has been tested. The "pole assignment" work, and the control works on the full non-linear system (at least up to $Re = 75$)

Ongoing developments

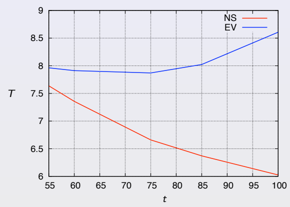
- Continue to analyse the *Re* dependence for this type of feedback control
- Test the control on systems with more unstable modes.

Introduction The optimal control problem Min-energy feedback control Results

N-S versus LST

Comparing results obtained with N-S and Linear Stability Theory.

period



Strouhal number