Stochastic approach to the receptivity problem

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Outline

- Introduction & Background
  - Receptivity problem
  - Motivation for stochastic approach
- Stochastic initial condition
  - By-pass transition
  - Realizability of optimal disturbances
  - Comparison with optimal IC
- Stochastic forcing
  - Uncorrelated forcing
  - Noise colouring
  - Results for boundary layers
- Conclusions

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Boundary layer transition

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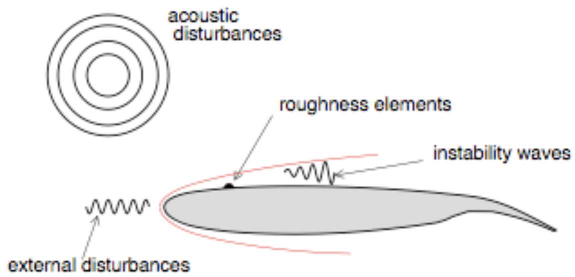
- Transition is caused by breakdown of growing disturbances inside the boundary layer
- Instability theory can accurately compute growth or decay of perturbations
- Initial amplitudes of unstable waves need to be estimated to capture transition location:
  Receptivity problem



Stochastic Approach to Receptivity

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Motivation

- Predict the disturbance level for initial and external conditions, expected or modeled
- Estimate the realizability of optimal disturbances
- Robustness of deterministic results

Use a stochastic approach assuming a statistic description of external perturbations

Problem formulation

- Linear discrete governing equations

$$\dot{q} = \mathcal{A}q + f\,u(t), \quad q(x, t=0) = q_0.$$

- Covariance of the flow defined as

$$P(x, x', t) = \mathrm{cov}(q(x,t), q(x',t)) = \langle qq^H \rangle$$

- Lyapunov equation for evolution of P

$$\dot{P} = \mathcal{A}P + P\mathcal{A}^H + M, \quad P(0) = P_0$$

P_0, the covariance of the initial condition, and M, covariance of the external forcing, are modeled.

$$P_0 = \langle q_0 q_0^H \rangle; \; M = \langle f f^H \rangle; \; \langle u(t_1)u(t_2) \rangle = \delta(t_1 - t_2)$$

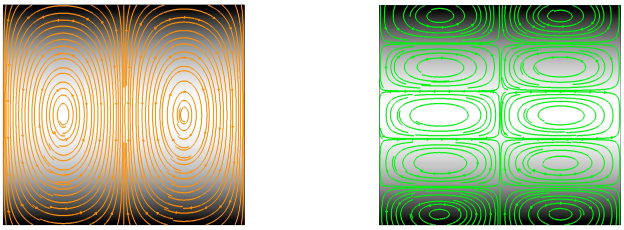
Stochastic initial value problem

Solution of the Lyapunov equation

$$\dot{P} = \mathcal{A}P + P\mathcal{A}^H$$

- Matrix equation: direct solution expensive $P(t) = e^{\mathcal{A}t} P_0 e^{\mathcal{A}^H t}$
- Use POD modes of P_0: $[\phi_i, \lambda_i] = eig(P_0 \mathcal{Q})$

$\mathcal{Q}$ defines the kinetic energy norm



$i = 1$ $\quad$ $i = 10$

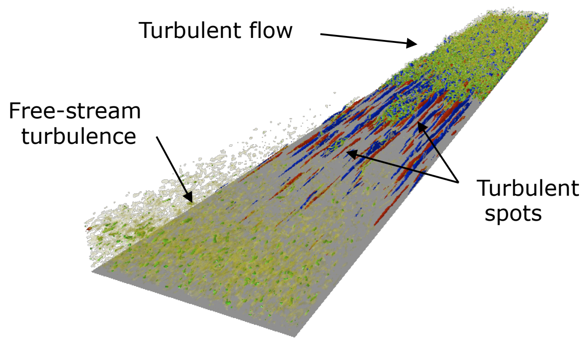
Solve the equation for the most energetic modes $\dot{\phi}_i = \mathcal{A}\phi_i$

Reconstruct the covariance matrix P of the disturbance further downstream $P = \sum_i \phi(x)\lambda_i\phi_i(x)^H$

Application to bypass transition

Consider boundary layer subject to free-stream turbulence

Correlations from spectrum of homogeneous isotropic turbulence



Disturbance description

Define covariance of initial free-stream perturbation:

Von Karman spectrum of homogeneous isotropic turbulence

$$E(k) = \frac{2}{3} \frac{a(kL)^4}{(b + (kL)^2)^{17/6}} Lq$$

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Fourier transform of two-point velocity correlation

$$< u_i u_j > = \frac{E(k)}{4\pi k^2} \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right)$$

Disturbance description

Covariance of initial free-stream perturbation

Smoothly reduced to zero at the boundary-layer edge

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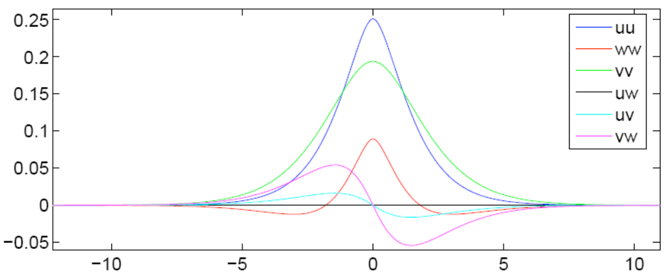


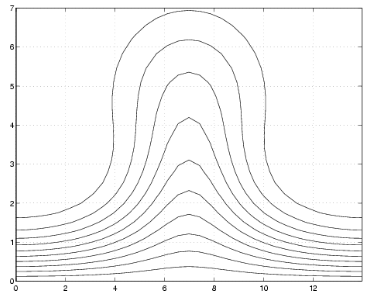
$$< u_i u_j > (y, y'; \alpha, \beta)$$

Problem non-homogenous wall-normal direction y

Fourier modes in the homogeneous directions (x,z)

Secondary instability of streaks

- Consider steady nonlinearly saturated streaks arising from optimally growing streamwise vortices at the leading edge



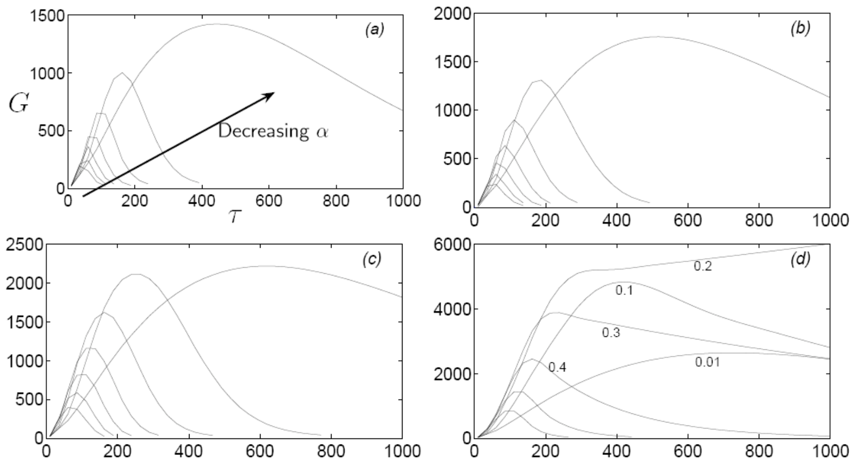
Cross-stream distribution of streamwise velocity

- Compute the largest possible energy amplification as function of time

Hœpffner, Brandt & Henningson (**JFM**, 2005)

Secondary transient growth

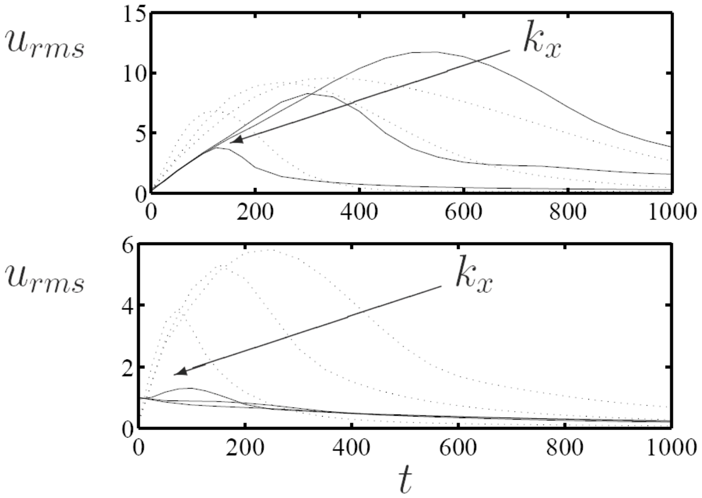
Sinuous modes



α =0.01, 0.1, 0.2, …, 0.6

(a),(b),(c),(d): Amplitudes: A=0.14, 0.2, 0.25, 0.29.

Response to stochastic initial conditions



Antisymmetric waves

Symmetric modes

Stochastic noise in the free stream u_{rms}: ———

Maximum possible growth $1/5\sqrt{E/E_0}$: ········

Hœpffner & Brandt (**Phys Fluid**, 2008)

Stochastic forcing

Stochastic forcing

Steady-state Lyapunov equation: $0 = \mathcal{A}P + P\mathcal{A}^H + M$

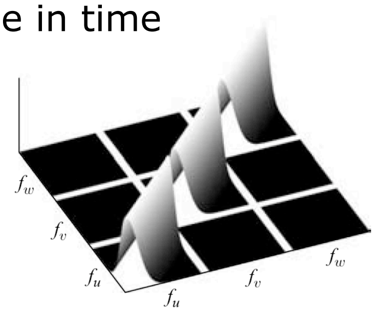
- Uncorrelated white noise $\quad M = \langle f f^H \rangle = \mathbf{I}$

  Farrel & Ioannou, Jovanovic & Bamieh

- Spatially correlated white noise in time

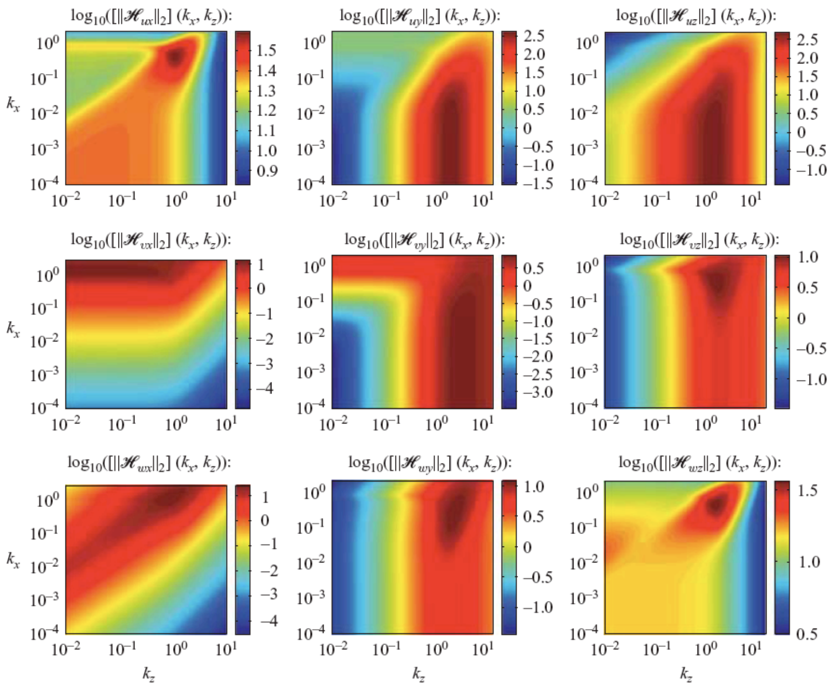
  Hoepffner et al, Chevalier et al

- Colored noise



Stochastic uncorrelated forcing

Component-wise analysis (Jovanovic & Bamieh 2005)



Stochastic forcing

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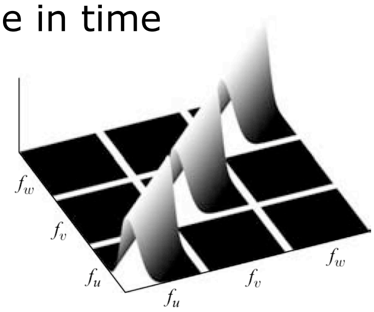


Stochastic time-correlated noise

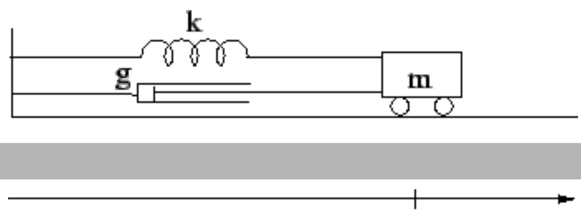
Lyapunov equation assumes white noise

$$\begin{pmatrix} \dot{q} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} \mathcal{A} & f \\ 0 & F \end{pmatrix} \begin{pmatrix} q \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix} u(t)$$

Time-correlated noise obtained by augmented system

F filter for white noise: determines the forcing features

2 x 2 system with complex conjugate eigenvalues

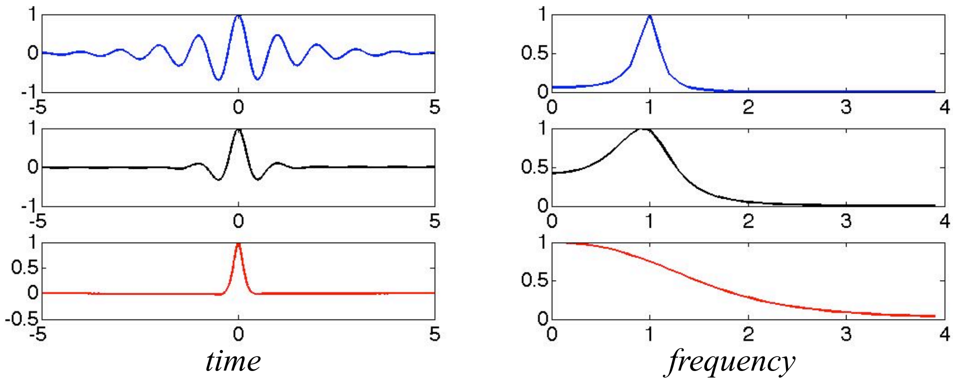


Noise coloring

Two DOF filter:

dominant frequency f=1

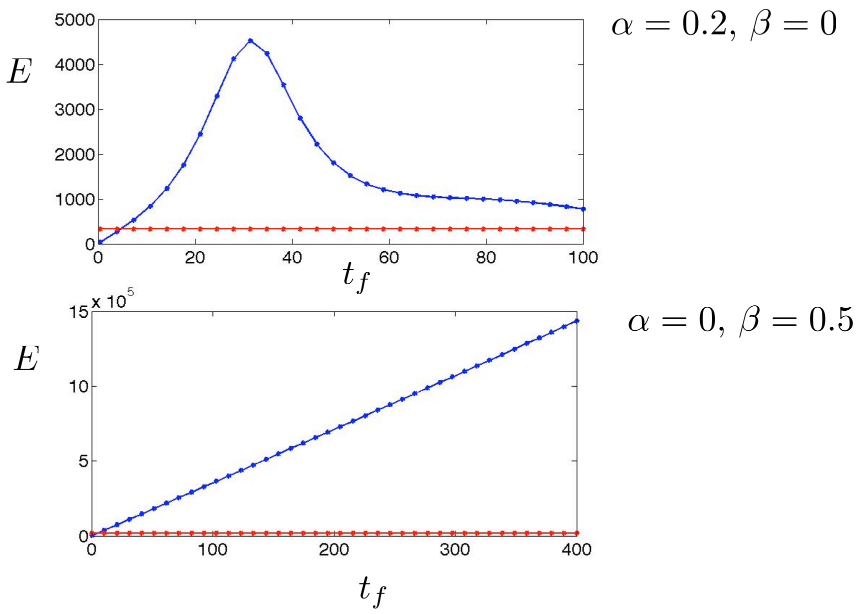
peak dominance d=6, 2, 0.5



More complicated model designed in similar way

Energy of flow response

- Comparison with uncorrelated forcing,
  parallel Blasius at Re=500



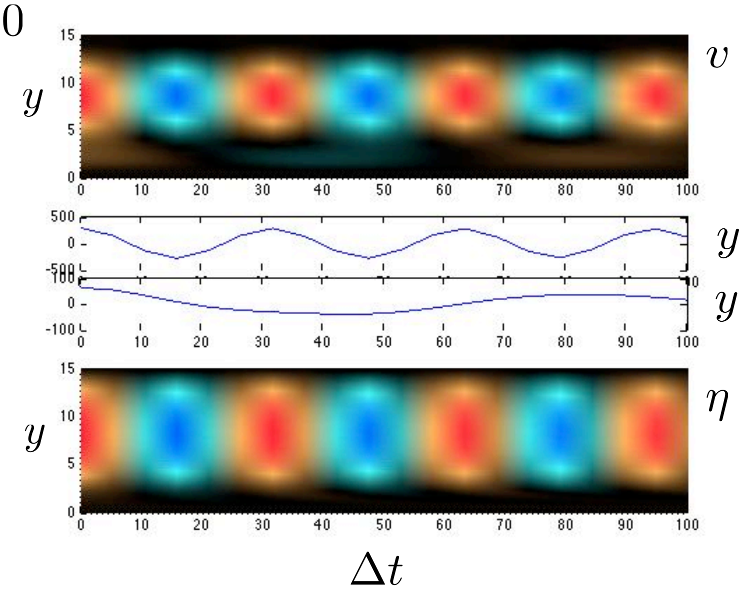
*Spatial- and time-correlated noise: y_f=3, y_d=5, t_d=2*t_f*

Correlation of flow response

- Two-time correlation from correlated forcing,
  parallel Blasius at Re=500

$\alpha = 0.2,\ \beta = 0$

$t_d = 30$



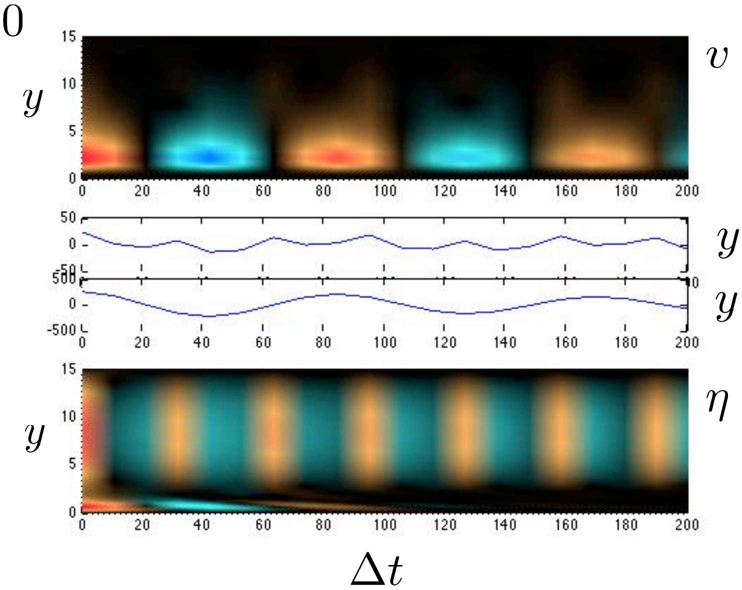
*Spatial- and time-correlated noise: y_f=3, y_d=5, td=30, t_d=2*t_f*

Correlation of flow response

- Two-time correlation from correlated forcing,
  parallel Blasius at Re=500

$\alpha = 0.2,\ \beta = 0$

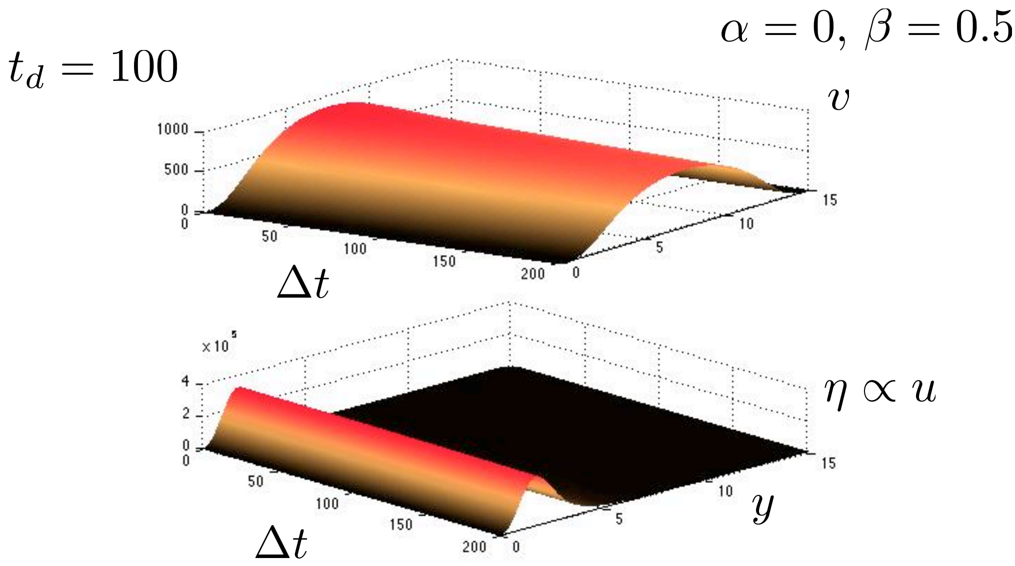
$t_d = 80$



*Spatial- and time-correlated noise: y_f=3, y_d=5, td=80, t_d=2*t_f*

Correlation of flow response

- Two-time correlation from correlated forcing, parallel Blasius at Re=500



Spatial- and time-correlated noise: $y_f=3$, $y_d=5$, $td=100$, $t_d=2*t_f$

Conclusions

- Introduction & Background
  - Relevance of the receptivity problem
- Stochastic initial condition
  - Bypass transition: ability of free-stream to trigger streak breakdown
  - Sinuous modes most receptive
- Stochastic forcing
  - Time-correlated noise
  - Different time and length scales excited

  - Parameter study

FLOW Graduate School

- Summer school on FLOW Control, June 29-July 3 2008
  (week after IUTAM Symposium Laminar-Turbulent Transition)

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Optimal control, Feedback Control, Model Reduction, Numerical Methods, Applications

Invited lecturers:
Clancy Rowley, Peter Schmid, Bernd Noack