

Transition and multistability in magnetoconvection

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Keywords: *Magnetoconvection, Instability, Dynamical systems, Transition to turbulence*

Summary:

Understanding the interplay between convection and magnetic fields has long been motivated in geophysical and astrophysical contexts for studying stellar interiors, liquid-metal planetary cores, and accretion disks. In recent years, however, industrial interest in magnetoconvection (MC) has surged, with applications extending to liquid-metal batteries, electromagnetic brakes in continuous casting, semiconductor crystal growth and liquid-metal cooling systems for fusion reactors. Traditional theoretical work in MC predominantly considers laterally periodic domains, a simplification that aligns with geo-astrophysical applications, aiming to effectively capture the system's behaviour on large spatial scales. However, the effects of sidewalls are intrinsic to industrial applications and many laboratory experiments, where convection often occurs in geometrically confined settings. In this thesis, we demonstrate that the presence of sidewalls, rarely considered in detail in MC, dramatically influences the instabilities and emergent nonlinear dynamics in this system, both close to onset and in the turbulent regime. In particular, we find that novel nonlinear instabilities associated with the sidewalls lead to multistability between flow states at identical operating conditions and are fundamental in forming the large-scale turbulent dynamics of the system. The mechanisms we reveal revise our convective understanding of turbulence in magnetoconvection, and potentially other strongly constrained systems, but also provide new pathways for efficiently controlling transport properties in future engineering devices.

Scientific summary:

We focus on liquid-metal (low Prandtl number Pr , and in the low magnetic Reynolds number limit) MC subjected to a strong vertical magnetic field B_0 . The thermal driving in the system and the strength of the constraining force provided by the magnetic field are quantified by the Rayleigh number (Ra) and the Hartmann number (Ha), respectively. The main theme of this work is to understand aspects of the transition to turbulence which are specific to the confined container geometries often encountered in engineering applications and experiments. Therefore, instead of assuming lateral periodicity of the domain as in traditional MC studies, we consider a cube with rigid electrically insulating boundaries, an idealisation of the weakly conducting walls typically encountered in practice. We have performed direct numerical simulations (DNS) of this flow in a cubic domain over a wide range of the (Ra, Ha) parameter space, and for orders of magnitude longer timescales than previous simulations in the literature. This allows us to examine the complex behaviour of this system across a variety of unexplored regimes. The simulations are complemented with theoretical and data-driven analysis to answer a number of fundamental questions.

Near-onset behaviour:

A sufficiently strong vertical magnetic field completely suppresses convection through the Lorentz force, leaving the flow in a motionless state where heat is transported purely by conduction. For a given Ha , the motionless state undergoes a linear instability in a laterally periodic layer upon increased Ra , leading to domain-filling arrays of convection rolls similar to non-magnetic Rayleigh–Bénard convection. However, when insulating sidewalls are introduced, the Lorentz force vanishes at the wall, allowing convection to be triggered more easily in thin layers near the sidewalls than in the bulk. This results in a different linear instability, with wall-localised onset modes, known as wall modes. This linear instability occurs at $Ra_{c,L}$, where stability is exchanged from the motionless state to the linear onset branch (LB) in a supercritical pitchfork bifurcation, shown for $Ha = 500$ in figure 1(a). The LB state features two thin roll structures along every sidewall, each with a two-layer wall-normal structure (figure 1b, LB). Having found the linear stability threshold for the motionless state, its nonlinear stability is then assessed. Two additional families of nonlinear equilibrium solutions have been identified: the subcritical branch (SB), with a single roll along each sidewall, and the mixed-symmetry branch (MB), which features one/two roll combinations on adjacent walls (see figure 1b, SB/MB). Remarkably, both SB and MB remain stable beneath the linear stability threshold ($Ra < Ra_{c,L}$), an effect which becomes stronger with Ha . This implies that the transition can occur subcritically, where finite-amplitude disturbances trigger the transition despite a linearly stable motionless state, similar to wall-bounded shear flows. Unlike supercritical transitions, subcritical transitions occur over a range of parameter values, depending on the system's noise, leading to hysteresis and large, unpredictable changes in the response. Importantly, this behaviour does not appear in non-magnetic convection or magnetoconvection

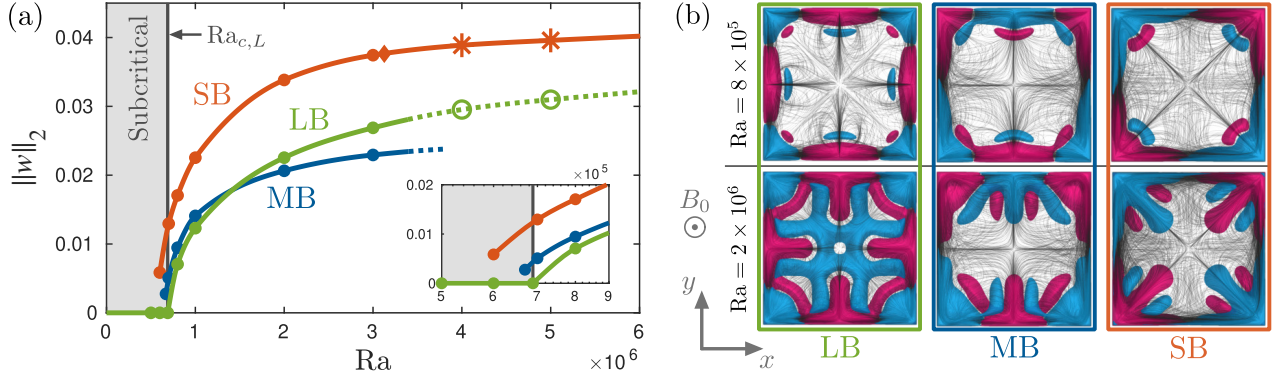


Figure 1: (a) Bifurcation diagram at $Ha = 500$ showing stable/unstable equilibria (filled/open markers), limit cycles (diamonds), invariant tori (asterisks) on the LB, SB and MB branches, using the norm of the time-averaged vertical velocity to distinguish solutions. The grey area represents $Ra < Ra_{c,L}$. (b) Vertical velocity isosurfaces (pink/blue) and streamlines (black) of solutions on the various branches shown from a top view at $Ha = 500$ at $Ra = 8 \times 10^5$ (top) and $Ra = 2 \times 10^6$ (bottom) with $w = \pm 0.005$ and ± 0.01 , respectively. Figure reproduced from [2].

in a laterally periodic layer, indicating that sidewalls entirely change the phenomenology of the transition from supercritical to subcritical.

Transition to turbulence:

Although the presence of subcritical solutions is fundamentally interesting, their relevance in the transition to chaos and in turbulence itself must be investigated. At $Ha = 500$, the LB and MB equilibria become unstable in the range $3 \times 10^6 < Ra \leq 4 \times 10^6$ (figure 1a). Above this threshold, initial conditions close to LB or MB exhibit long transients before being attracted to the SB branch, which undergoes the transition to chaos. Spectral analysis reveals that SB undergoes a series of Hopf bifurcations, first forming a stable limit cycle and then a stable invariant 2-torus (figure 1a). Further increasing Ra leads to chaotic dynamics, and therefore, the system follows a quasiperiodic route to chaos in the spirit of the Ruelle–Takens–Newhouse theorem which predicts more generally that 3-tori are unlikely to be structurally stable. At $Ha = 1000$, SB again undergoes a Hopf bifurcation, but then exhibits complex homoclinic behaviour reminiscent of Shilnikov dynamics before becoming turbulent [1]. However, the transition is still subcritical and occurs along the SB branch. In all cases, the solutions exhibit strong wall mode activity. As Ra is increased from the near-onset regime, the wall modes grow into the bulk of the domain, forming protrusions due to nonlinear effects, which begin to flap back and forth as the solutions become time-dependent. These wall modes continuously reorient the bulk flow, setting its large-scale structure [1]. As the system approaches turbulence, vertically orientated columnar structures contribute to chaotic bulk dynamics, however, the wall modes are still seen to be prominent. For example, at $Ha = 1000$ with $Ra = 2 \times 10^7$, wall mode protrusions penetrate far into the domain, shedding large-scale structures into the bulk (figure 2a). This is an important observation since wall modes are often suggested to be confined within thin Shercliff boundary layers on the sidewalls, and therefore are expected to play a negligible role in the bulk dynamics as the system transitions to turbulence. However, the effect of wall modes on turbulence is clearly evident in the mean flow: unlike in typical supercritical transitions, where the first linear instability typically imprints the turbulent flow (e.g. Taylor vortices in turbulent Taylor–Couette flow or convection rolls in turbulent Rayleigh–Bénard convection), here the turbulent flow inherits the spatial organisation and symmetry of the subcritical SB wall mode equilibria, not LB (figure 2b-d). This implies that understanding the details of the subcriticality and the associated nonlinear wall modes, which can only be reliably revealed through carefully designed DNS or experiments, is necessary to fundamentally understand the organising flow structures in the turbulent regime. Therefore, the results imply that we must change the way we think about the transition to turbulence and the formation of large-scale structures in convective and potentially other nonlinear pattern-forming systems to account for these subtleties, even when linear stability theory predicts a supercritical transition. Furthermore, the clear influence of sidewalls here suggests that magnetoconvection in closed-vessel experiments is fundamentally different to the laterally unbounded geo-astronomical flows they often intend to replicate in nature.

Multiple turbulent states:

In the turbulent regime, a second attractor has been discovered by increasing the magnetic field strength from non-magnetic turbulent convection [3]. The resulting flow state, which coexists with the cellular state developed through the subcritical wall-mode route to turbulence, features a single large-scale turbulent roll. Although the two states have been observed in experimentally in separate regimes depending on the magnetic field, we show here that they actually coexist over a wide range of parameter space. This marks a qualitatively new type of multi-stability in magnetoconvection, where different turbulent states can be obtained from different initial conditions for

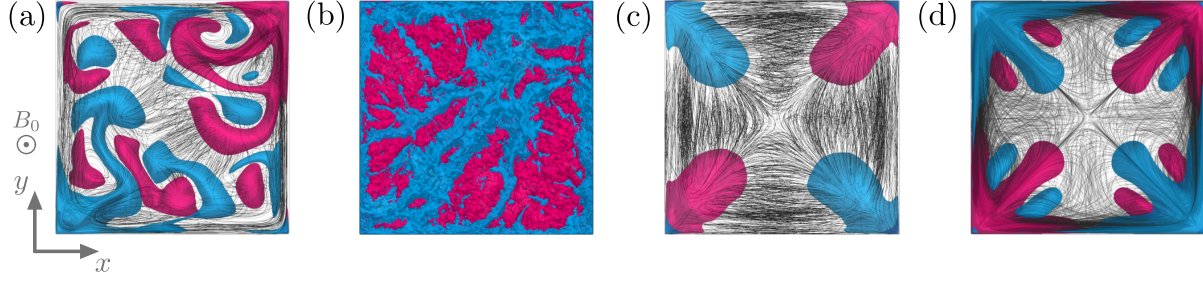


Figure 2: Vertical velocity isosurfaces (pink/blue) and streamlines (black) for at (a) $Ha = 1000$, $Ra = 2 \times 10^7$, (b/c) $Ha = 500$, $Ra = 10^9$ (instantaneous/mean), and (d) $Ha = 500$, $Ra = 2 \times 10^6$ (SB). Figure reproduced from [1,2].

a given (Ra, Ha) [3]. Each solution has a distinct mean flow and transport properties. Using a sparsity-promoting variant of dynamic mode decomposition, we analyse the spatio-temporal dynamics of these states to understand their differences beyond globally averaged quantities. We find that the cellular state relies on a hierarchy structures with progressively smaller-scale features, linked to the bifurcations in the route to chaos, whereas the roll state is dominated by a single large-scale roll mode coupled to disordered small-scale dynamics. As the system is pushed to higher Ra , the two relative attractors collide in state-space, resulting in one transitioning to a chaotic saddle, exhibiting finite-lifetime transient chaos. The preferred globally attracting flow structure at low Ha is the roll state, whereas the cellular state appears to be preferred at high Ha . Therefore, by increasing and decreasing Ra in the turbulent regime, a hysteresis loop can be formed, leading to transitions between the two states. This discovery of coexisting, structurally distinct turbulent states not only has major theoretical implications, but also practical applications. In particular, different initial conditions or the hysteresis effect can be exploited to control and optimise the behaviour of industrial devices. For example, in the weakly turbulent regime, the cellular state can provide up to 20% more efficient heat transport under identical operating conditions.

Heat transport model:

To better understand how the heat transport of the system varies across the Ra - Ha parameter space, we have developed an analytic model with a single adjustable parameter that accurately captures the dimensionless heat transport (Nusselt number, Nu) from the buoyancy-dominated regime at large Ra to the Lorentz-dominated regime at large Ha [4,5]. The model is constructed under the assumption that the transition between these regimes is controlled by the relative thickness of the viscous and thermal boundary layers. This assumption allows for the heat transport across (Ra, Ha) to be collapsed to a master curve, which itself depends on a single scalar parameter [4]. This parameter s , depends on the control parameters in a way that is determined by our theory, that is, $s = s(x)$ where $x = \log[Ha^{-1/\gamma}Ra]$ where γ is the scaling exponent in the buoyancy dominated regime, obtained by Grossmann-Lohse theory. A simple ansatz for $s(x)$ provides an explicit analytical approximation for $Nu = Nu(Ra, Ha)$ with a single free parameter which depends on Pr [5]. The model has been validated for quasistatic magnetoconvection and at finite magnetic Reynolds numbers using data from existing DNS and experiments [4,5], and is therefore well-suited to be incorporated into complex multi-physics models frequently used in applied sciences and engineering applications.

Selected publications and total research output:

The research presented in this thesis has resulted in five publications, with one more in preparation. The results have been showcased through presentations and posters at 14 national and international conferences. Papers [1,4] have been highlighted through a focal point article in *Physik Journal*. Five key publications are listed below:

1. McCormack, M., Teimurazov, A., Shishkina, O., and Linkmann, M., 2023, Wall mode dynamics and transition to chaos in magnetoconvection with a vertical magnetic field, *J. Fluid Mech.*, 975, R2. ([Link](#)).
2. McCormack, M., Teimurazov, A., Shishkina, O., and Linkmann, M., 2025, Subcritical transition and multi-stability in liquid metal magnetoconvection with sidewalls, *J. Fluid Mech.*, 1023, R1. ([Link](#)).
3. McCormack, M., Teimurazov, A., Shishkina, O., and Linkmann, M., 2026, Multiple turbulent states in liquid metal magnetoconvection, *In preparation*.
4. Teimurazov, A., McCormack, M., Linkmann, M., and Shishkina, O., 2024, Unifying heat transport model for the transition between buoyancy-dominated and Lorentz-force-dominated regimes in quasistatic magnetoconvection, *J. Fluid Mech.*, 980, R3. ([Link](#)).
5. McCormack, M., Teimurazov, A., Shishkina, O., and Linkmann, M., 2025, Heat transport model for the transition between scaling regimes in quasistatic and fluctuating magnetoconvection, *Int. J. Heat Mass Transf.*, 241, 126641, ([Link](#)).